One-Dimensional, Steady-State Conduction without Thermal Energy Generation

Chapter Three

Sections 3.1 through 3.4

Methodology of a Conduction Analysis

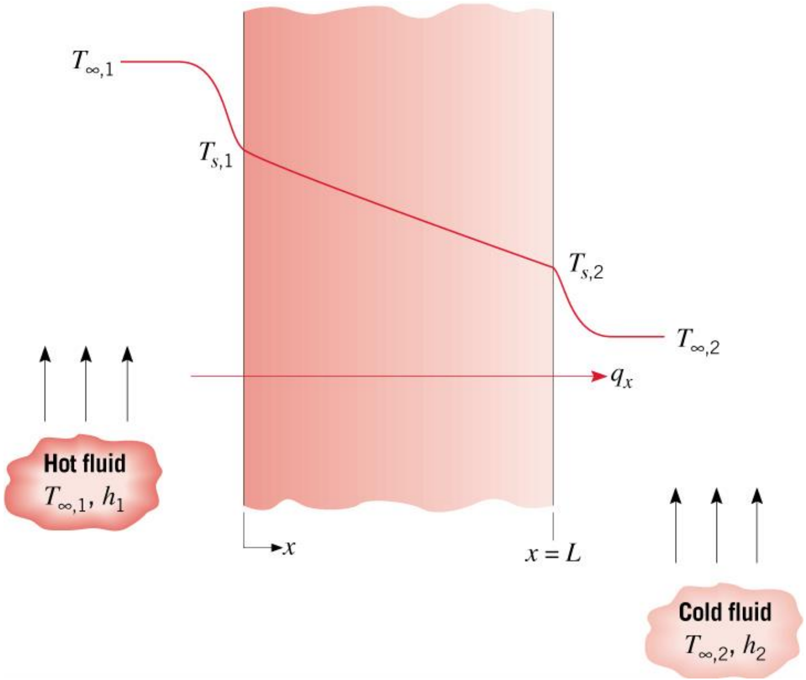
- Specify appropriate form of the heat equation.
- Solve for the temperature distribution.
- Apply Fourier's law to determine the heat flux.

Simplest Case: One-Dimensional, Steady-State Conduction with No Thermal Energy Generation.

- Common Geometries:
  - The Plane Wall: Described in rectangular (*x*) coordinate. Area perpendicular to direction of heat transfer is constant (independent of *x*).
  - The Tube Wall: Radial conduction through tube wall.
  - The Spherical Shell: Radial conduction through shell wall.

The Plane Wall

- Consider a plane wall between two fluids of different temperature:



- Heat Equation:

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0 \qquad (3.1)$$

- Implications:

  Heat flux $\left(q''_x\right)$ is independent of x.

  Heat rate $\left(q_x\right)$ is independent of x.

- Boundary Conditions: $T(0) = T_{s,1}, \quad T(L) = T_{s,2}$

- Temperature Distribution for Constant k:

$$T(x) = T_{s,1} + \left(T_{s,2} - T_{s,1}\right)\frac{x}{L} \qquad (3.3)$$

- Heat Flux and Heat Rate:

$$q''_x = -k\frac{dT}{dx} = \frac{k}{L}\left(T_{s,1} - T_{s,2}\right) \quad (3.5)$$

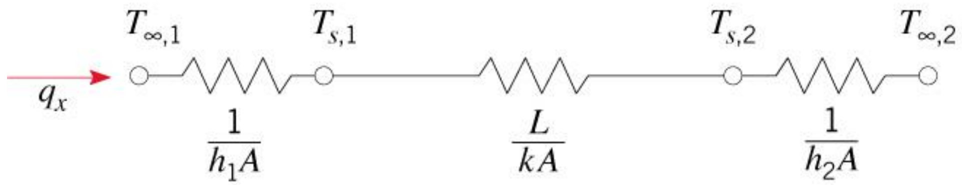
$$q_x = -kA\frac{dT}{dx} = \frac{kA}{L}\left(T_{s,1} - T_{s,2}\right) \quad (3.4)$$

- Thermal Resistances $\left(R_t = \frac{\Delta T}{q}\right)$ and Thermal Circuits:

Conduction in a plane wall: $R_{t,\text{cond}} = \frac{L}{kA}$ (3.6)

Convection: $R_{t,\text{conv}} = \frac{1}{hA}$ (3.9)

Thermal circuit for plane wall with adjoining fluids:



$$R_{\text{tot}} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A} \quad (3.12)$$

$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{R_{\text{tot}}} \quad (3.11)$$

- Thermal Resistance for Unit Surface Area:

$$R''_{t,\text{cond}} = \frac{L}{k} \qquad R''_{t,\text{conv}} = \frac{1}{h}$$

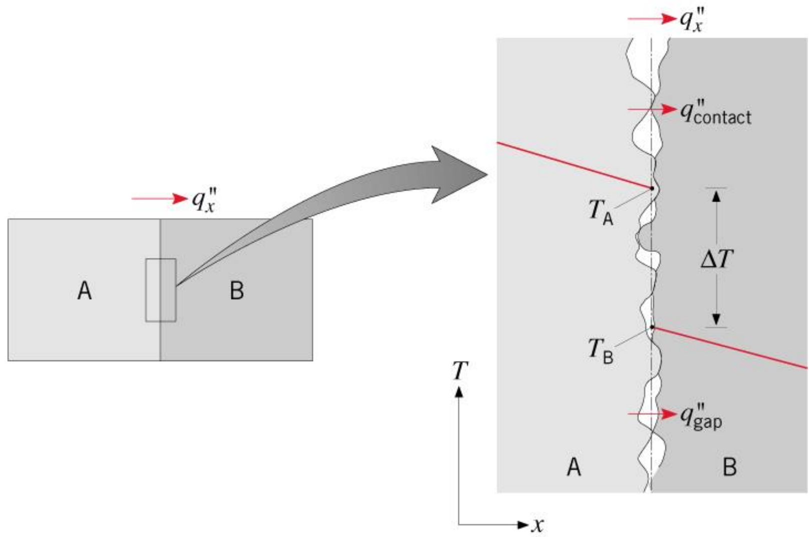
Units: $R_t \leftrightarrow \text{K/W} \qquad R''_t \leftrightarrow \text{m}^2 \cdot \text{K/W}$

- Radiation Resistance:

$$R_{t,\text{rad}} = \frac{1}{h_r A} \qquad R''_{t,\text{rad}} = \frac{1}{h_r}$$

$$h_r = \varepsilon\sigma\left(T_s + T_{\text{sur}}\right)\left(T_s^2 + T_{\text{sur}}^2\right) \tag{1.9}$$

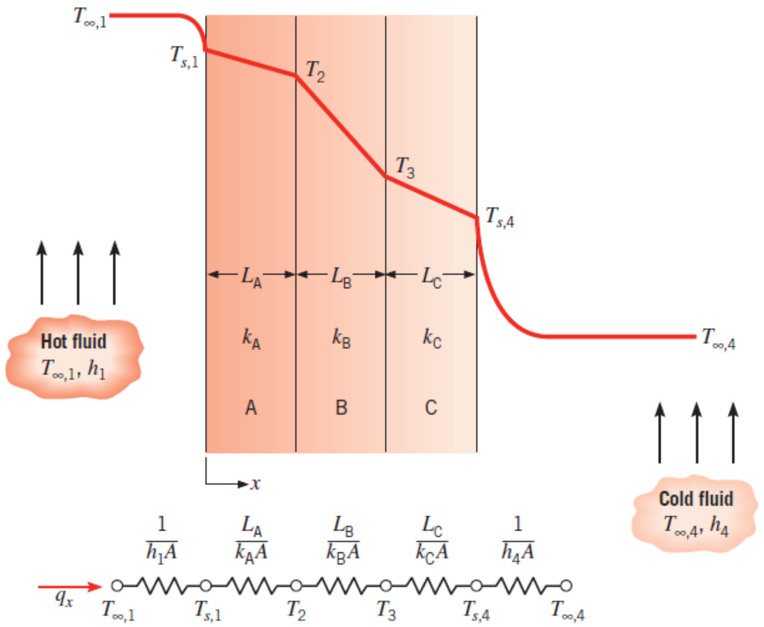
- Contact Resistance:



$$R''_{t,c} = \frac{T_A - T_B}{q''_x} \qquad R_{t,c} = \frac{R''_{t,c}}{A_c}$$

Values depend on: Materials A and B, surface finishes, interstitial conditions, and contact pressure (Tables 3.1 and 3.2)

- Composite Wall with Negligible Contact Resistance:



$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{\sum R_t} \tag{3.14}$$

For the temperature distribution shown, $k_A > k_B < k_C$.

$$\sum R_t = R_{tot} = \frac{1}{A}\left[\frac{1}{h_1} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{h_4}\right] = \frac{R''_{tot}}{A}$$

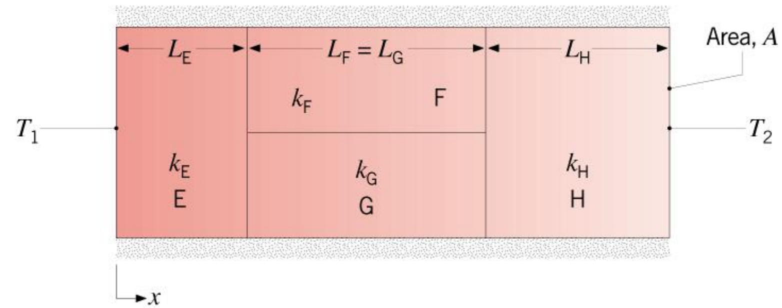
- Overall Heat Transfer Coefficient (U) :

A modified form of Newton's law of cooling to encompass multiple resistances to heat transfer.

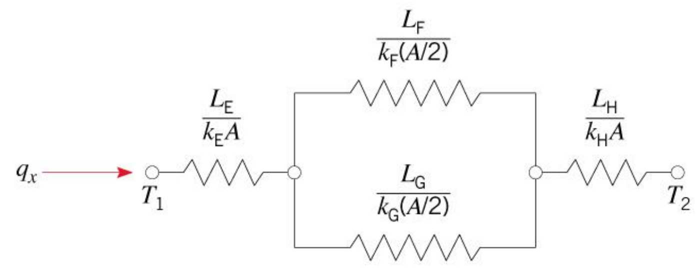
$$q_x = UA\Delta T_{overall} \tag{3.17}$$

$$R_{tot} = \frac{1}{UA} \tag{3.19}$$

- Series – Parallel Composite Wall:

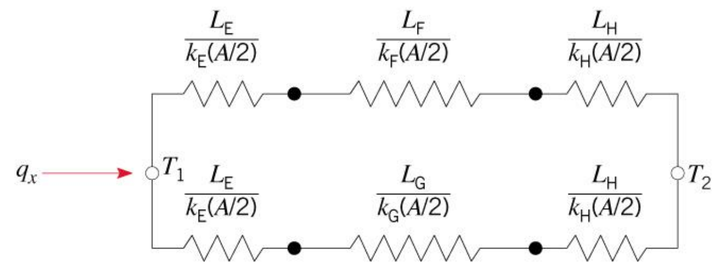


Assuming isothermal surfaces perpendicular to *x*-direction.



(*a*)

Assuming adiabatic surfaces parallel to *x*-direction.

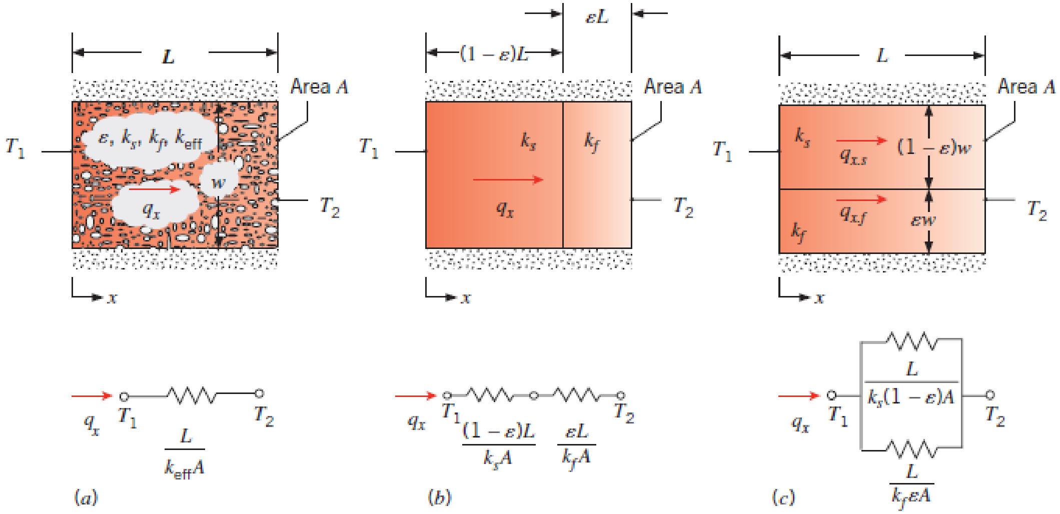


(*b*)

- Note departure from one-dimensional conditions for $k_F \neq k_G$.

- Circuits based on assumption of isothermal surfaces normal to *x* direction or adiabatic surfaces parallel to *x* direction provide approximations for q_x .

- Porous Media



- Saturated media consist of a solid phase and a single fluid phase.
- Unsaturated media consist of solid, liquid, and gas phases.

- The effective thermal conductivity of a saturated medium depends on the solid (s) material, its porosity ε, its morphology, as well as the interstitial fluid (f) (Fig.a).

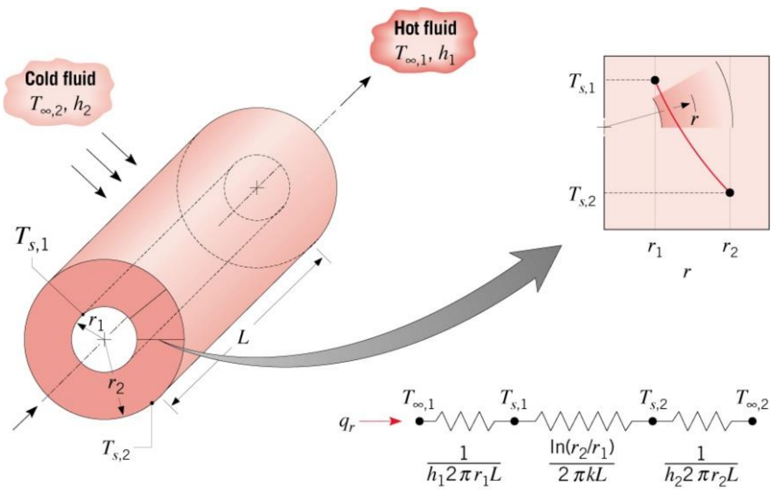
$$q_x = \frac{k_{eff}A}{L}(T_1 - T_2) \qquad (3.21)$$

- The value of k_{eff} may be *bracketed* by describing the medium with a series resistance analysis (Fig. b) and a parallel resistance analysis (Fig.c).
- The value of k_{eff} may be *estimated* by

$$k_{eff} = \left[\frac{k_f + 2k_s - 2\varepsilon(k_s - k_f)}{k_f + 2k_s + \varepsilon(k_s - k_f)}\right]k_s \qquad (3.25)$$

$$\varepsilon \leq 0.25$$

The Tube Wall



- Heat Equation:

$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right)=0 \qquad (3.28)$$

What does the form of the heat equation tell us about the variation of q_r with r in the wall?

Is the foregoing conclusion consistent with the energy conservation requirement?

How does q_r'' vary with r?

- Temperature Distribution for Constant k:

$$T(r)=\frac{T_{s,1}-T_{s,2}}{\ln(r_1/r_2)}\ln\left(\frac{r}{r_2}\right)+T_{s,2} \qquad (3.31)$$

- Heat Flux and Heat Rate:

$$q_r'' = -k\frac{dT}{dr} = \frac{k}{r\ln\left(r_2/r_1\right)}\left(T_{s,1}-T_{s,2}\right) \qquad [\mathrm{W/m^2}]$$

$$q_r' = 2\pi r q_r'' = \frac{2\pi k}{\ln\left(r_2/r_1\right)}\left(T_{s,1}-T_{s,2}\right) \qquad [\mathrm{W/m}]$$

$$q_r = 2\pi r L q_r'' = \frac{2\pi L k}{\ln\left(r_2/r_1\right)}\left(T_{s,1}-T_{s,2}\right) \qquad [\mathrm{W}] \qquad (3.32)$$

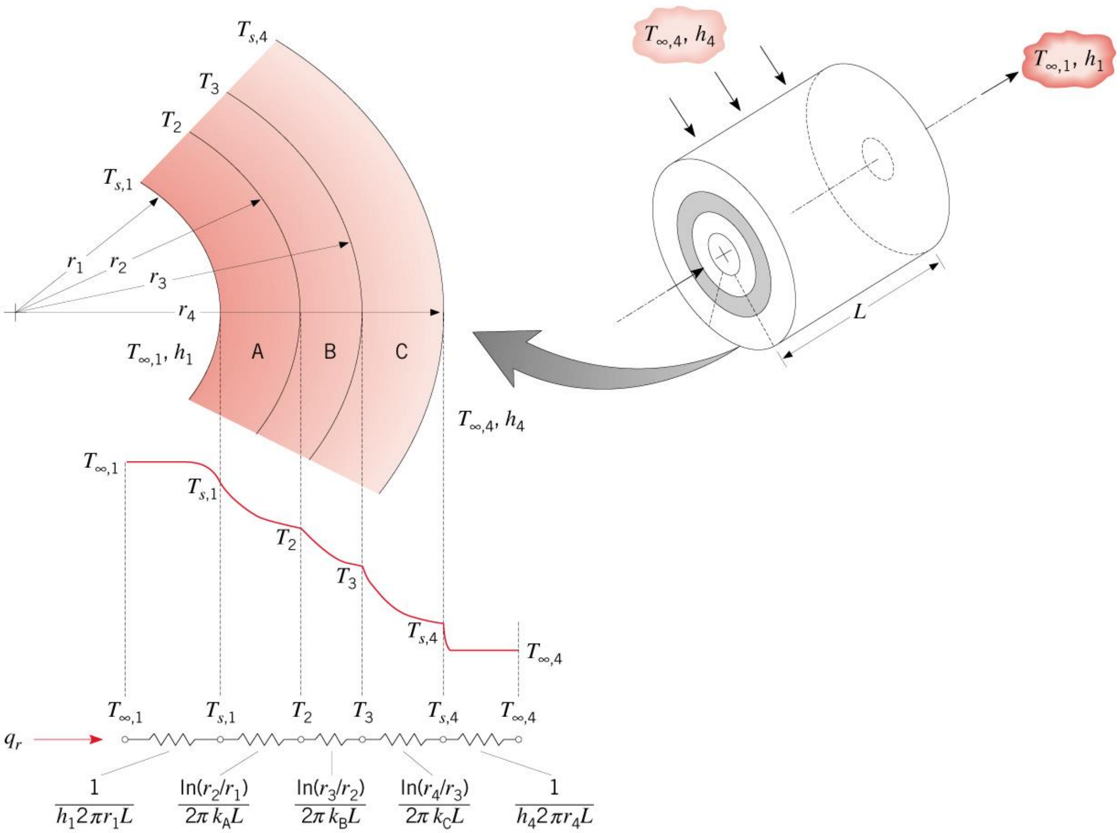
- Conduction Resistance:

$$R_{t,\text{cond}} = \frac{\ln\left(r_2/r_1\right)}{2\pi L k} \qquad [\mathrm{K/W}] \qquad (3.33)$$

$$R'_{t,\text{cond}} = \frac{\ln\left(r_2/r_1\right)}{2\pi k} \qquad [\mathrm{m\cdot K/W}]$$

Why doesn’t a surface area appear in the expressions for the thermal resistance?

- Composite Wall with Negligible Contact Resistance



$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{R_{\text{tot}}}$$

$$= UA\left(T_{\infty,1} - T_{\infty,4}\right) \quad (3.35)$$

For the temperature distribution shown, $k_A > k_B > k_C$.

Note that

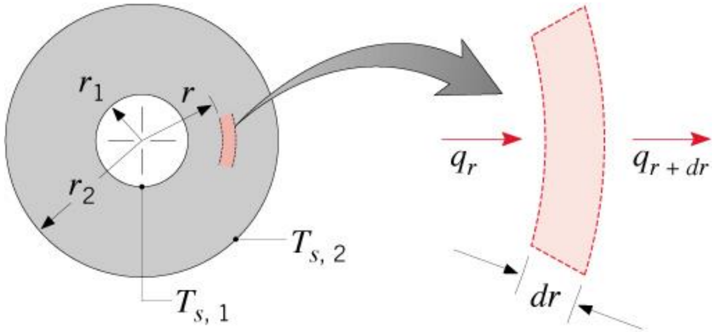
$$UA = R_{\text{tot}}^{-1}$$

is a constant independent of radius,

but U itself is tied to specification of an interface.

$$U_i = \left(A_i R_{\text{tot}}\right)^{-1} \quad (3.37)$$

Spherical Shell



- Heat Equation for Constant k:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)=0$$

What does the form of the heat equation tell us about the variation of q_r with r? Is this result consistent with conservation of energy?

How does q_r'' vary with r?

- Temperature Distribution:

$$T(r)=T_{s,1}-\left(T_{s,1}-T_{s,2}\right)\frac{1-\left(r_1/r\right)}{1-\left(r_1/r_2\right)}$$

- Heat Flux, Heat Rate and Thermal Resistance:

$$q''_r = -k\frac{dT}{dr} = \frac{k}{r^2\left[(1/r_1)-(1/r_2)\right]}\left(T_{s,1}-T_{s,2}\right)$$

$$q_r = 4\pi r^2 q''_r = \frac{4\pi k}{(1/r_1)-(1/r_2)}\left(T_{s,1}-T_{s,2}\right) \qquad (3.40)$$

$$R_{t,\text{cond}} = \frac{(1/r_1)-(1/r_2)}{4\pi k} \qquad (3.41)$$

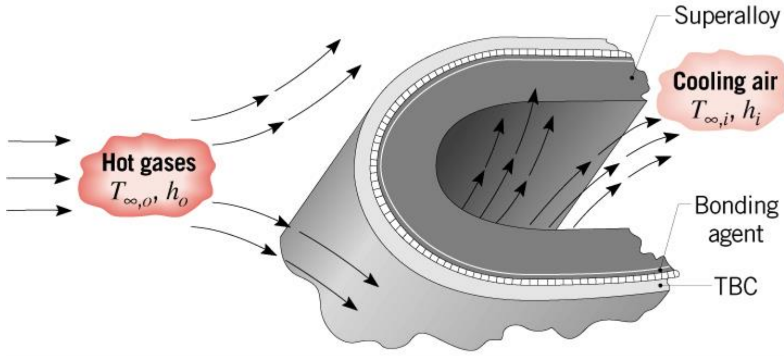
- Composite Shell:

$$q_r = \frac{\Delta T_{\text{overall}}}{R_{\text{tot}}} = UA\Delta T_{\text{overall}}$$

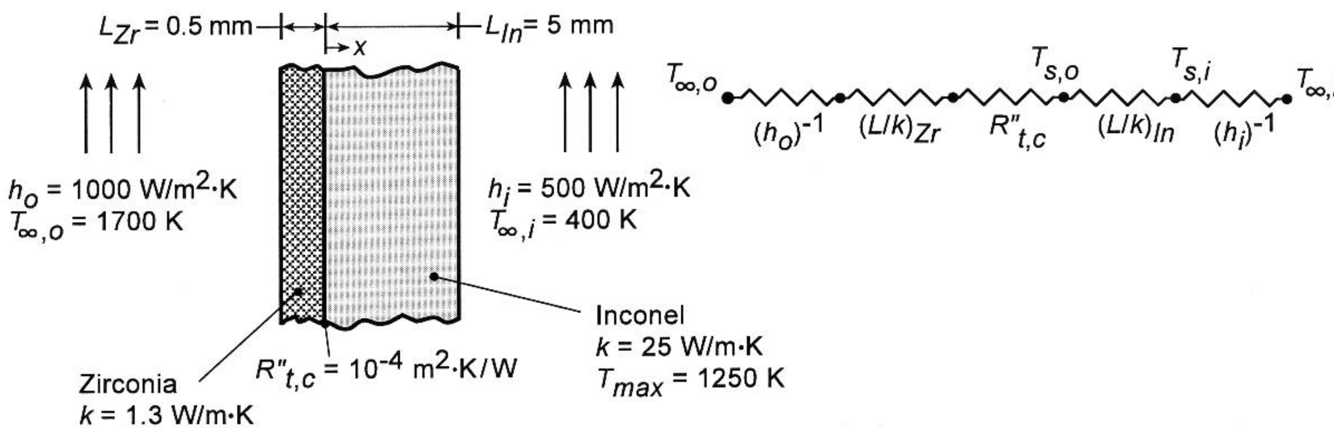
$$UA = R_{\text{tot}}^{-1} \leftrightarrow \text{Constant}$$

$$U_i = \left(A_i R_{\text{tot}}\right)^{-1} \leftrightarrow \text{Depends on } A_i$$

Problem 3.23: Assessment of thermal barrier coating (TBC) for protection of turbine blades. Determine maximum blade temperature with and without TBC.



SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional, steady-state conduction in a composite plane wall, (2) Constant properties, (3) Negligible radiation.

ANALYSIS: For a unit area, the total thermal resistance with the TBC is

$$R''_{\text{tot},w} = h_o^{-1} + (L/k)_{\text{Zr}} + R''_{t,c} + (L/k)_{\text{In}} + h_i^{-1}$$

$$R''_{\text{tot},w} = \left(10^{-3} + 3.85\times10^{-4} + 10^{-4} + 2\times10^{-4} + 2\times10^{-3}\right)\text{m}^2\cdot\text{K/W} = 3.69\times10^{-3}\ \text{m}^2\cdot\text{K/W}$$

With a heat flux of

$$q''_w = \frac{T_{\infty,o} - T_{\infty,i}}{R''_{\text{tot},w}} = \frac{1300\,\text{K}}{3.69\times10^{-3}\,\text{m}^2\cdot\text{K/W}} = 3.52\times10^5\ \text{W/m}^2$$

the inner and outer surface temperatures of the Inconel are

$$T_{s,i(\text{w})} = T_{\infty,i} + (q''_w/h_i) = 400\,\text{K} + \left(\frac{3.52\times10^5\,\text{W/m}^2}{500\,\text{W/m}^2\cdot\text{K}}\right) = 1104\,\text{K}$$

$$T_{s,o(\text{w})} = T_{\infty,i} + \left[(1/h_i) + (L/k)_{\text{In}}\right]q''_w$$

$$= 400\,\text{K} + \left(2\times10^{-3} + 2\times10^{-4}\right)\text{m}^2\cdot\text{K/W}\left(3.52\times10^5\,\text{W/m}^2\right) = 1174\,\text{K} \qquad <$$

Without the TBC,

$$R''_{\text{tot},\text{wo}} = h_o^{-1} + (L/k)_{\text{In}} + h_i^{-1} = 3.20 \times 10^{-3}\ \text{m}^2 \cdot \text{K/W}$$

$$q''_{\text{wo}} = \left(T_{\infty,o} - T_{\infty,i}\right)/R''_{\text{tot,wo}} = 4.06\times 10^5\ \text{W/m}^2$$

The inner and outer surface temperatures of the Inconel are then

$$T_{s,i(\text{wo})} = T_{\infty,i} + \left(q''_{wo}/h_i\right) = 1212\ \text{K}$$

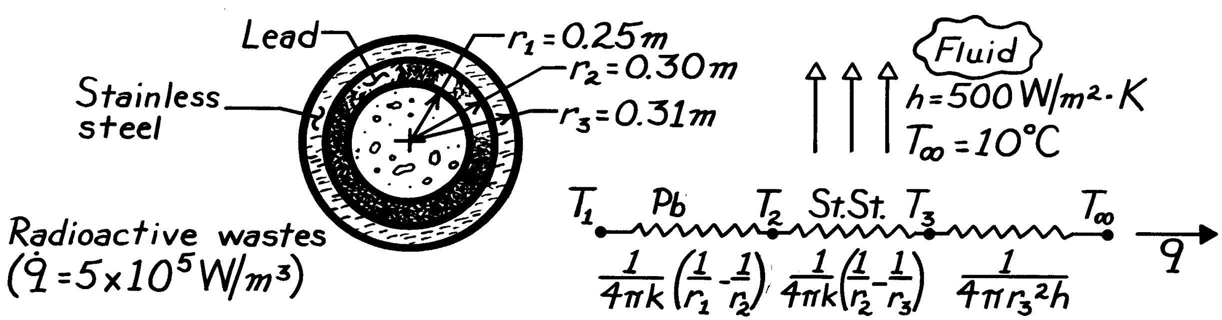
$$T_{s,o(\text{wo})} = T_{\infty,i} + \left[(1/h_i) + (L/k)_{\text{In}}\right] q''_{wo} = 1293\ \text{K}$$ <

Use of the TBC facilitates operation of the Inconel below T_{max} = 1250 K.

COMMENTS: Since the durability of the TBC decreases with increasing temperature, which increases with increasing thickness, limits to its thickness are associated with reliability considerations. In this application, thermal contact resistance is desirable.

Problem 3.56: Suitability of a composite spherical shell for storing radioactive wastes in oceanic waters.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Steady-state conditions, (3) Constant properties at 300K, (4) Negligible contact resistance.

PROPERTIES: Table A-1, Lead: k = 35.3 W/m·K, MP = 601 K; St.St.: k = 15.1 W/m·K.

ANALYSIS: From the thermal circuit, it follows that

$$q = \frac{T_1 - T_\infty}{R_{tot}} = \dot{q}\left[\frac{4}{3}\pi\, r_1^3\right]$$

The thermal resistances are:

$$R_{Pb} = \left[1/\left(4\pi \times 35.3\ \text{W/m}\cdot\text{K}\right)\right]\left[\frac{1}{0.25\text{m}} - \frac{1}{0.30\text{m}}\right] = 0.00150\ \text{K/W}$$

$$R_{St.St.} = \left[1/\left(4\pi \times 15.1\ \text{W/m}\cdot\text{K}\right)\right]\left[\frac{1}{0.30\text{m}} - \frac{1}{0.31\text{m}}\right] = 0.000567\ \text{K/W}$$

$$R_{conv} = \left[1/\left(4\pi \times 0.31^2\text{m}^2 \times 500\ \text{W/m}^2\cdot\text{K}\right)\right] = 0.00166\ \text{K/W}$$

$$R_{tot} = 0.00372\ \text{K/W}$$

The heat rate is then

$$q = 5\times10^5\ \text{W/m}^3\left(4\pi/3\right)\left(0.25\text{m}\right)^3 = 32,725\ \text{W}$$

and the inner surface temperature is

$$T_1 = T_\infty + R_{tot}q = 283\ \text{K} + 0.00372\ \text{K/W}\left(32{,}725\ \text{W}\right)$$

$$= 405\ \text{K} < \text{MP} = 601\ \text{K}$$ <

Hence, from the thermal standpoint, the proposal is adequate.

COMMENTS: In fabrication, attention should be given to maintaining a good thermal contact. A protective outer coating should be applied to prevent long term corrosion of the stainless steel. In this application, thermal contact resistance is undesirable.